

V.A. Miransky*

Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

I.A. Shovkovy*

School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA

(August 23, 2001)

We describe a class of relativistic models with matter in which spontaneous breakdown of continuous symmetries leads to a lesser number of Nambu-Goldstone bosons than that required by the Goldstone theorem. This class, in particular, describes the dynamics of the kaon condensate in the color-flavor locked phase of high density QCD. We describe the spectrum of low energy excitations in this dynamics and show that, despite the presence of a condensate and gapless excitations, this system is not a superfluid.

11.30.Qc, 12.38.Aw, 21.65.+f

The Goldstone theorem is a cornerstone of the phenomenon of spontaneous breakdown of continuous global symmetries. It is applicable both to relativistic field theories with exact Lorentz symmetry [1] and to most condensed matter systems [2], though there is no this symmetry. However, there is an important difference between these two cases. While in Lorentz invariant systems, the Goldstone theorem is universally valid, it is not so in condensed matter systems. For example, it does not apply to condensed matter systems with long range interactions [2]. From the technical viewpoint, the difference is connected with a kinetic term, and derivative terms, in general, in a Lagrangian density: while their form is severely restricted by the Lorentz symmetry, it is much more flexible in systems where this symmetry is absent.

In this letter, we describe the phenomenon of spontaneous symmetry breaking of continuous symmetries with an abnormal number of Nambu-Goldstone bosons taking place in a class of interesting models *without* long range interactions. Here by “abnormal”, we understand that the number of gapless NG bosons is less than the number of the generators in the coset space G/H , where G is an initial symmetry and H is a symmetry group of the ground state (vacuum). On the other hand, as we will see, the degeneracy of the vacuum retains conventional: it is described by transformations connected with all the generators from the coset space. It is noticeable that this class of models describes a recently suggested [3–5] scenario with a kaon condensate in the color-flavor locked (CFL) phase of high density QCD [6].

We will illustrate this phenomenon in a toy model with the following Lagrangian density:

$$\mathcal{L} = (\partial + i\mu)\Phi^\dagger(\partial - i\mu)\Phi - v^2\partial_i\Phi^\dagger\partial_i\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2, \quad (1)$$

where $\Phi^T = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2, \tilde{\phi}_1 + i\tilde{\phi}_2)$ is a complex doublet field and v is a velocity parameter ($v < 1$). The chemical potential μ is provided by external conditions (to be specific, we take $\mu > 0$). The above Lagrangian density is invariant under global $SU(2) \times U(1)$. The $SU(2)$ will be treated as the isospin group I and the $U(1)$ will be associated with hypercharge Y . The electric charge Q is $Q = I_3/2 + Y$. This model describes the essence of the dynamics of the kaon condensate [4] (see below).

When $\mu < m$, it is straightforward to derive the tree level spectrum of the physical degrees of freedom. To this end, we switch to the momentum space. Then, the quadratic part of the above lagrangian takes the following form:

$$\mathcal{L}^{(2)}(\omega, q) = \frac{1}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \mathcal{M} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 \end{pmatrix} \tilde{\mathcal{M}} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}, \quad (2)$$

where

$$\mathcal{M} = \tilde{\mathcal{M}} = \begin{pmatrix} \omega^2 + \mu^2 - m^2 - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 + \mu^2 - m^2 - v^2q^2 \end{pmatrix}. \quad (3)$$

The dispersion relations of the particles is determined from the equation $\text{Det}(\mathcal{M}) = 0$. Explicitly, this equation reads:

$$[(\omega - \mu)^2 - m^2 - v^2q^2] [(\omega + \mu)^2 - m^2 - v^2q^2] = 0, \quad (4)$$

i.e., the particle’s dispersion relations are:

$$\omega_1 = \tilde{\omega}_1 = \pm \sqrt{m^2 + v^2 q^2} + \mu, \quad (5)$$

$$\omega_2 = \tilde{\omega}_2 = \pm \sqrt{m^2 + v^2 q^2} - \mu. \quad (6)$$

Of course, the positive (negative) values of energy correspond to creation (annihilation) of particles.

Therefore the particle spectrum contains two doublets with masses $m + \mu$ and $m - \mu$, respectively. More precisely, the first doublet can be identified as (K_-, \bar{K}_0) and the second one as (K_+, K_0) . The crucial point is that, because of the chemical potential, the masses of particles and their antiparticles split up. This is intimately connected with the fact that C , CP , and CPT symmetries are explicitly broken in this system. In second quantized theory, the complex field Φ^T describes creation and annihilation of the (K_-, \bar{K}_0) and (K_+, K_0) doublets, respectively, though their masses are now different.

By studying the potential of the above model in tree approximation, we could check that the perturbative vacuum becomes a local maximum when $\mu > m$ [see Eq. (2) with $\omega = q = 0$]. At this point, the system experiences an instability with respect to forming a condensate. In the new phase, a vacuum expectation value, $\tilde{\phi}_0$, of the field Φ occurs,

$$\Phi = \begin{pmatrix} 0 \\ \tilde{\phi}_0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \tilde{\phi}_1 + i\tilde{\phi}_2 \end{pmatrix}. \quad (7)$$

By requiring that the new vacuum is a minimum of the potential, we derive

$$\tilde{\phi}_0^2 = \frac{\mu^2 - m^2}{2\lambda} \quad (8)$$

for the vacuum expectation value of the field. At this vacuum, the initial $SU(2) \times U(1)$ spontaneously breaks down to $U(1)_Q$.

At the new vacuum, the quadratic form reads:

$$\mathcal{L}_{new}^{(2)}(\omega, q) = \frac{1}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \mathcal{M} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 \end{pmatrix} \tilde{\mathcal{M}} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} \quad (9)$$

where

$$\mathcal{M} = \begin{pmatrix} \omega^2 + \mu^2 - m^2 - 2\lambda\phi_0^2 - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 + \mu^2 - m^2 - 2\lambda\phi_0^2 - v^2q^2 \end{pmatrix}, \quad (10)$$

and

$$\tilde{\mathcal{M}} = \begin{pmatrix} \omega^2 + \mu^2 - m^2 - 6\lambda\phi_0^2 - v^2q^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 + \mu^2 - m^2 - 2\lambda\phi_0^2 - v^2q^2 \end{pmatrix}. \quad (11)$$

For the first two states, the dispersion relations are (henceforth we retain only positive eigenvalues)

$$\omega_{1,2} = \sqrt{\mu^2 + v^2q^2} \pm \mu. \quad (12)$$

For the other two states, the dispersion relation is given by the following formula:

$$\tilde{\omega}_{1,2} = \sqrt{3\mu^2 - m^2 + v^2q^2 \pm \sqrt{(3\mu^2 - m^2)^2 + 4\mu^2v^2q^2}}. \quad (13)$$

All four dispersion relations are shown in Fig. 1. One of the relations in Eq. (12), i.e.,

$$\omega_2 = \sqrt{\mu^2 + v^2q^2} - \mu, \quad (14)$$

and one of the relations in Eq. (13), i.e.,

$$\tilde{\omega}_2 = \sqrt{3\mu^2 - m^2 + v^2q^2 - \sqrt{(3\mu^2 - m^2)^2 + 4\mu^2v^2q^2}}, \quad (15)$$

correspond to NG bosons (see Fig. 1). Indeed, in the far infrared region, these relations take the following form:

$$\omega_2 \simeq \frac{v^2q^2}{2\mu}, \quad (16)$$

$$\tilde{\omega}_2 \simeq \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} vq. \quad (17)$$

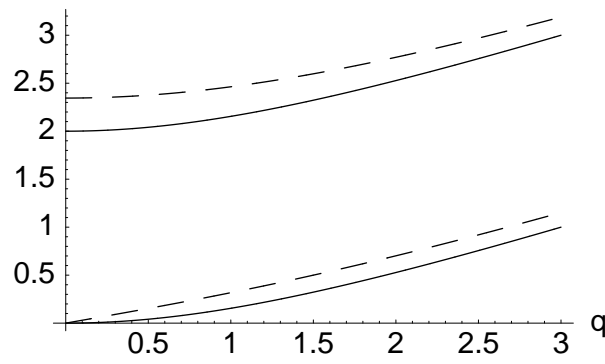


FIG. 1. Dispersion relations of four particles in the broken phase of the model in Eq. (1). The solid and dashed lines represent the relations in Eqs. (12) and (13), respectively. All quantities are given in units of μ .

Therefore, there are two NG bosons in the system. It comes as a real surprise. Indeed, since here the initial $SU(2) \times U(1)$ spontaneously breaks down to $U(1)_Q$, one should expect the existence of three NG bosons. Where is the third one?!

To understand better the situation at hand, it is instructive to consider the quadratic part of the potential around the vacuum with the condensate. For the two sets of the fields, the corresponding quadratic forms are read from $-\mathcal{M}$ and $-\tilde{\mathcal{M}}$ where $\omega = q = 0$ is substituted. The eigenvalues of such matrices are

$$\xi_{1,2} = \tilde{\xi}_2 = 0, \quad (18)$$

$$\tilde{\xi}_1 = 2(\mu^2 - m^2). \quad (19)$$

Therefore, we see that the potential part of the action has three flat directions in the broken phase, as it should be. These three directions would correspond to three NG bosons, related to the three broken generators of the original $SU(2) \times U(1)$. The first order derivative terms in the kinetic part of the action, however, prevent the appearance of one charged, K_- , gapless mode. This fact is intimately connected with the point that the presence of the chemical potential leads to mass splitting between K_- and K_+ .

Another noticeable fact is that the energy ω_2 of the K_+ gapless mode in Eq. (16) is proportional to q^2 rather than to q . This implies that, despite the presence of gapless modes, the Landau criterion for superfluidity fails in this model.¹ Therefore, this system is not a superfluid.

This toy model illustrates a rather general phenomenon. While in Lorentz invariant systems with spontaneous breakdown of continuous symmetries the degeneracy of the potential fixes the number of NG bosons being equal to the number of generators $N_{G/H}$ in the coset space G/H , in systems with a broken Lorentz symmetry the number of NG bosons can be lesser. The latter is connected with a form of the kinetic term which can include first order derivative terms. In the systems under consideration, the chemical potential (leading here to such first order derivative terms) splits up masses of charged particles and their antiparticles *both* of which would be NG bosons otherwise. On the other hand, a neutral NG boson, a partner of a field with a nonzero vacuum expectation value, always survives. Therefore, the number of the NG bosons reduces by N_{ch} , where N_{ch} is the number of charged particle-antiparticle would be NG pairs. In the present model $N_{G/H} = 3$ and $N_{ch} = 1$, and as a result there are only two (one neutral and one charged) NG bosons.

The choice of the simple model in Eq. (1) was not accidental in this paper. It describes the essence of a much more complicated dynamics of spontaneous symmetry breaking related to the kaon condensation in the color-flavor locked (CFL) phase of dense quark matter [4]. Let us turn to it.

We start by repeating briefly the analysis of Ref. [4]. In the chiral limit, the ground state of the three flavor dense quark matter corresponds to the CFL phase [6]. The original $SU(3)_c \times SU(3)_L \times SU(3)_R$ symmetry of the microscopic action breaks down to global “locked” $SU(3)_{c+L+R}$ subgroup. The low-energy action for the NG bosons could be derived in the asymptotic limit [8], where the parameters of the effective theory could be calculated from the microscopic theory. The current quark masses break the original chiral symmetry of the model explicitly. As a

¹Recall that, according to the Landau criterion [7], superfluidity takes place for velocities $v < v_c$, where $v_c = \min_{i,q}(\omega_i(q)/q)$ is the minimum taken over all excitation branches and all values of momentum q . Therefore even the presence of a single branch with $\omega \sim q^2$ implies that $v_c = 0$.

result, the NG bosons obtain nonzero masses, becoming pseudo-NG bosons. Their dynamics could still be described by the low-energy action which, for sufficiently small current quark masses, could be derived from the microscopic theory [8]. By making use of an auxiliary “gauge” symmetry, it was suggested in Ref. [4] that the low-energy action of Refs. [9,8] should be modified by adding a time-like covariant derivative to the action of the composite field.

By neglecting a chemical potential of the electric charge, the low energy effective Lagrangian of Ref. [4] (in Minkowski space) reads

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{f_\pi^2}{4} \text{Tr} [\nabla_0 \Sigma \nabla_0 \Sigma^\dagger - v_\pi^2 \partial_i \Sigma \partial_i \Sigma^\dagger] + \frac{1}{2} [(\partial_0 \eta')^2 - v_{\eta'}^2 (\partial_i \eta')^2] \\ & + 2A \left[\det(M) \text{Tr} \left(M^{-1} \Sigma e^{\sqrt{\frac{2}{3}} \frac{i}{f_{\eta'}} \mathcal{I} \eta'} \right) + h.c. \right], \end{aligned} \quad (20)$$

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \frac{M M^\dagger}{2p_F} \Sigma - i \Sigma \frac{M^\dagger M}{2p_F}, \quad (21)$$

where M is a quark mass matrix chosen to be diagonal, i.e., $M = \text{diag}(m_u, m_d, m_s)$. By definition, \mathcal{I} is a unit matrix in the flavor space, and Σ is a unitary matrix field which describes the octet of the NG bosons, transforming under the chiral $SU(3)_L \times SU(3)_R$ group as follows:

$$\Sigma \rightarrow U_L \Sigma U_R^\dagger, \quad \text{where } (U_L, U_R) \in SU(3)_L \times SU(3)_R. \quad (22)$$

In Eq. (20), we also took into account η' field which couples to the octet when the quark masses are nonzero. The NG boson, related to breaking the baryon number, was omitted, however. Its dynamics is not affected much by the quark masses.

One should notice from the definition of the covariant derivative in Eq. (21) that the a combination of the quark mass matrices $\mu_{eff} = M M^\dagger / 2p_F$ produces effective chemical potentials for different flavor charges. These chemical potentials are of dynamical origin, and they are unavoidable. In addition, one could also add a nonzero chemical potential of the electric charge $\mu_e \neq 0$. Without losing the generality, we omit it in our discussion below.

The presense of the effective chemical potential μ_{eff} has far reaching consequences. In particular, if the mismatch of the quark masses of different flavors is large enough [e.g., $m_s \gtrsim (\Delta^2 m_u)^{1/3}$], the perturbative CFL vacuum becomes unstable with respect to a kaon condensation.

The new ground state is determined by a “rotated” vacuum expectation value of the Σ field,

$$\Sigma_\alpha(\pi) \equiv \exp(i\alpha\lambda^6) \exp(i\pi_A \lambda^A) \simeq \exp(i\alpha\lambda^6) \left(1 + \frac{i\pi_A \lambda^A}{f_\pi} - \frac{\pi_A \pi_B \lambda^A \lambda^B}{2f_\pi^2} + \dots \right), \quad (23)$$

where α is determined by requiring that the corresponding vacuum is a global maximum of the potential energy. In the simplest case with $m_u = m_d$, we derive

$$\cos \alpha = \frac{4cp_F^2 m_u (m_s + m_u)}{f_\pi^2 (m_s^2 - m_u^2)^2} < 1, \quad (24)$$

where (see Ref. [8])

$$c = \frac{3\Delta^2}{2\pi^2} \quad \text{and} \quad f_\pi^2 = \frac{21 - 8 \ln 2}{36} \frac{\mu^2}{\pi^2}. \quad (25)$$

The ground state with the kaon condensation which is determined by Eq. (23) breaks the $SU(2) \times U(1)_Y$ symmetry of the effective action (20) down to $U(1)$. This is exactly the symmetry breaking pattern that we encountered in the model (1).

The derivation of the dispersion relations around the vacuum with the kaon condensation involves rather tedious calculations. Qualitatively, though, such a derivation is similar to that in the model in Eq. (1). Additional difficulties come from more complicated particle mixing. By omitting the details, we present the results.

With the choice of the vacuum expectation value in Eq. (23), the original 9 degrees of freedom (π_A with $A = 1, \dots, 8$ and η') group into two decoupled sets: $(\pi_1, \pi_2, \pi_4, \pi_5)$ and $(\pi_3, \pi_6, \pi_7, \pi_8 \equiv \eta, \eta')$. The first set of states contains all charged degrees of freedom, while the second set contains neutral ones.

The dispersion relations for the charged degrees of freedom $(\pi_1, \pi_2, \pi_4, \pi_5)$ are straightforward to derive. Our analysis shows that there is only 1 gapless NG boson in this set. In the far infrared region $q \rightarrow 0$, its explicit dispersion relation reads

$$\omega \simeq \frac{q^2}{3\mu \cos \alpha (1 + \cos \alpha)}, \quad (26)$$

where $\mu = (m_s^2 - m_u^2)/2p_F$. The analysis of the dispersion relations of the other set is more difficult. But it is also straightforward to show that there is only 1 gapless NG boson there as well. Moreover, for $q \rightarrow 0$, the explicit dispersion relation reads

$$\omega \simeq \frac{\sin \alpha}{\sqrt{2 - 7 \cos^2 \alpha + 9 \cos^4 \alpha}} \frac{q}{\sqrt{3}}. \quad (27)$$

We see that, as in the case of the simple model in Eq. (1), there are only two gapless NG bosons, despite the fact that there are three broken symmetry generators in the phase with the kaon condensate. One should note, however, that as in the model in Eq. (1), the effective potential obtained from (20) has the required three flat directions. It is the first order derivative terms that are responsible for giving a nonzero mass to one of the charged bosons. The corresponding opposite charge partner remains a gapless NG boson. It is crucial to note, though, that its dispersion relation behaves as $\omega \sim q^2$ for $q \rightarrow 0$. This implies that the criterion of superfluidity is not satisfied in the phase of the dense quark matter with kaon condensation.

As has been argued in Refs. [4,5], there is a good chance that the phase with kaon condensate may exist in a core of compact stars. Since the spectrum of low energy excitations in this phase derived in the present paper is very specific, implying in particular that the matter is not superfluid, it could play an important role in detecting the phase with kaon condensate.

In conclusion, in this paper a novel phenomenon of spontaneous symmetry breaking with an abnormal number of NG bosons was described. It admits a simple and clear interpretation. We expect that there exist wide applications of this phenomenon that deserve further study.

ACKNOWLEDGMENTS

The work of I.A.S. was supported by the U.S. Department of Energy Grant No. DE-FG02-87ER40328.

* On leave of absence from Bogolyubov Institute for Theoretical Physics, 252143, Kiev, Ukraine.

- [1] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127**, 965 (1962).
- [2] N.N. Bogolyubov, Physica **26**, 1 (1960); R.V. Lange, Phys. Rev. **146**, 301 (1966).
- [3] T. Schafer, Phys. Rev. Lett. **85**, 5531 (2000) [nucl-th/0007021];
- [4] P. F. Bedaque and T. Schafer, hep-ph/0105150.
- [5] D. B. Kaplan and S. Reddy, hep-ph/0107265.
- [6] M. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B **537**, 443 (1999) [hep-ph/9804403].
- [7] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2*, (Pergamon, 1980) pp. 88-93.
- [8] D. T. Son and M. A. Stephanov, Phys. Rev. D **61**, 074012 (2000) [hep-ph/9910491]; *ibid.* **62**, 059902 (2000) [hep-ph/0004095].
- [9] R. Casalbuoni and R. Gatto, Phys. Lett. B **464**, 111 (1999) [hep-ph/9908227].